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Pervasive Algebras of Analytic Functions

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We characterize those open U in the sphere such that $A(U)$ is complex-pervasive, and those such that $\operatorname{Re} A(U)$ is real-pervasive. Pervasive means, roughly, that the uniform closure on each proper closed subset of bdy U is the space of all continuous functions (to $\mathbb{C}$ or $\mathbb{R}$, as the case may be).

1. INTRODUCTION

Let X be a compact Hausdorff topological space and $C(X, \mathbb{C})$ (respectively, $C(X, \mathbb{R})$) the Banach algebra of all continuous complex-valued (respectively, real-valued) functions on X endowed with the uniform norm. A *function space* S on X is a closed subspace of $C(X, \mathbb{C})$. We denote by $\operatorname{clos}_{C(E, \mathbb{C})} S$ the closure in $C(E, \mathbb{C})$ of the function space S, where E is a closed subset of X. Similarly, we denote by $\operatorname{clos}_{C(E, \mathbb{R})} S$ the closure in $C(E, \mathbb{R})$ of the *real subspace* S of $C(X, \mathbb{R})$.

Let Y be a closed subset of X. A function space S on X is said to be *complex pervasive* on Y if $\operatorname{clos}_{C(E, \mathbb{C})} S = C(E, \mathbb{C})$ whenever E is a proper non-empty closed subset of Y. Similarly, a real subspace S of $C(X, \mathbb{R})$ is said to be *real pervasive* on Y if $\operatorname{clos}_{C(E, \mathbb{R})} S = C(E, \mathbb{R})$ whenever E is a proper non-empty closed subset of Y.

Let U be an open subset of the Riemann sphere $\hat{\mathbb{C}}$ and denote by bdy U its topological boundary. In this paper we consider the case when

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$X = \hat{\mathbb{C}}$, $Y = \text{bdy } U$ and S coincides with the algebra $A(U)$ of all complex-valued functions continuous on $\hat{\mathbb{C}}$ and analytic on U, or with Re $A(U)$, the space of real parts of elements of $A(U)$.

Obviously, if $A(U)$ is complex pervasive on bdy U then Re $A(U)$ is real pervasive on bdy U. Easy examples such as a pair of disjoint discs show that the converse is false.

A uniform algebra A, $A \subset C(X, \mathbb{C})$ is said to be *Dirichlet* on X if Re A is dense in $C(X, \mathbb{R})$ [5]. Thus Re $A(U)$ is real pervasive on bdy U if and only if $\text{clos}_{C(E, \mathbb{R})} A(U)$ is Dirichlet on E whenever E is a proper closed subset of bdy U.

The term pervasive was introduced by Hoffman and Singer in 1960 [7]. They studied (complex) pervasive uniform algebras, motivated by the relationship with maximal uniform algebras. For the algebras $A(U)$, they established that $A(U)$ is complex pervasive on bdy U if U is connected and $N \setminus U$ has positive area whenever N is a neighbourhood of a boundary point of U. This condition is, as we shall see, far from necessary.

In 1971, Gamelin and Garnett characterized those U for which $A(U)$ is Dirichlet on bdy U [6]. This result is deep. It is necessary that each component of U be simply-connected. Given that, the condition that $A(U)$ be Dirichlet is rather abstractly characterized by the pointwise bounded density of $A(U)$ in $H^\infty(U)$, and more concretely by a condition involving continuous analytic capacity, α. This condition may be expressed as follows. Let us say that the point $a \in \mathbb{C}$ is a *GG-point for* U if

$$\liminf_{r \downarrow 0} \frac{\alpha(\mathbb{U}(a, r) \setminus U)}{r} = 0,$$

where $\mathbb{U}(a, r)$ denotes the open disc with center a and radius r.

The Gamelin–Garnett Theorem. *Let $U \subset \hat{\mathbb{C}}$ be open, and suppose each component of U is simply-connected. Then $A(U)$ is Dirichlet on* bdy U *if and only if there are no GG-points for U on* bdy U.

Remark 1.1. Each GG-point on bdy U for U is an *inner* boundary point of U, i.e. it is not on the boundary of any component of the complement of clos U.

Since real pervasiveness may be re-expressed in terms of Dirichlicity of the algebras $A_E = \text{clos}_{C(E, \mathbb{R})} A(U)$, it is tempting to suppose that the Gamelin–Garnett Theorem settles the matter. This is not so, since A_E is not an $A(U)$ (nor is it one of the other algebras considered by Gamelin and Garnett in their paper). However, it is probable that the result of Gamelin and Garnett can be extended to all the so called T-invariant algebras (see below), with suitable modification, and the algebras A_E are T-invariant, so that one expects that real pervasiveness may be expressed in term of capacities associated to the A_E's. In fact, however, we shall see that a more direct approach may be used, employing the Gamelin–Garnett Theorem as it stands, and yielding a relatively simple and readily checked condition for real pervasiveness.

The real pervasiveness of spaces of harmonic functions on Euclidean spaces was studied by Netuka in [8]. He showed that if the open set $U \subset \mathbb{R}^d$ is bounded and connected, and bdy $U =$ bdy clos U, then the space of functions continuous on clos U and harmonic on U is real pervasive on bdy U. The present investigation was prompted by the question, whether, when $d = 2$, the space of harmonic functions could be replaced by the space Re $A(U)$ in this result. Realizing that the answer was yes, we proceeded to investigate the necessity of the conditions on U, and eventually were led to a complete characterization of the real pervasiveness of Re $A(U)$ and of the complex pervasiveness of $A(U)$.

In Section 2, we consider the case when U has *inessential boundary points*, i.e. points that are removable singularities for all elements of $A(U)$ (cf. Definition 2.1). This case reduces rather easily to classical facts.

In Section 3, we consider the case of connected U with *essential* boundary. This is perhaps the most natural situation, and we show that in it $A(U)$ is always complex pervasive on bdy U.

In Section 4, we consider general U. We give a complete characterization of complex pervasiveness in topological terms. This is not possible for real pervasiveness. We give a complete characterization involving continuous analytic capacity. This section is rather more technical and deeper than the rest of the paper relying as it does not only on the result of Gamelin and Garnett, but on Davie's deep result that characterizes the equality of two closed T-invariant algebras in term of the respective capacities associated to the algebras.

2. INESSENTIAL BOUNDARY POINTS

Given a compact Hausdorff topological space X, the dual space $C(X, \mathbb{C})^*$ of $C(X, \mathbb{C})$ will be identified with the space of complex Borel regular measures on X and it will be denoted by $M(X, \mathbb{C})$. Similarly $C(X, \mathbb{R})^*$ will be identified with the space of real Borel regular measures on X and denoted by $M(X, \mathbb{R})$. We regard $M(X, \mathbb{R})$ as a subset of $M(X, \mathbb{C})$. The (closed) support of a measure $\mu \in M(X, \mathbb{C})$ will be denoted by spt μ.

For a set $S \subset C(X, \mathbb{C})$ and a measure $\mu \in M(X, \mathbb{C})$ we write $\mu \perp S$, and say μ *annihilates* S, if $\int f \, d\mu = 0$ whenever $f \in S$.

As remarked in [3], one readily sees that a subspace $S \subset C(X, \mathbb{C})$ is complex pervasive (respectively a subspace $S \in C(X, \mathbb{R})$ is real pervasive) if and only if each nontrivial measure $\mu \in M(X, \mathbb{C})$ (respectively $M(X, \mathbb{R})$)

which annihilates S has spt $\mu = X$. Putting it in another way, S is complex pervasive (respectively, real pervasive) if and only if the conditions, $\mu \in M(X, \mathbb{C})$ (respectively, $M(X, \mathbb{R})$), $\mu \perp S$ and spt $\mu \subsetneq X$ imply that $\mu = 0$.

DEFINITION 2.1. Let a be a point in bdy U. We say that a is an *$A(U)$-inessential boundary point* if there exists $r > 0$ such that the inclusion map

$$A(U \cup \mathbb{U}(a, r)) \to A(U)$$

is surjective (and hence bijective), that is all functions in $A(U)$ extend analytically to $\mathbb{U}(a, r)$.

The *$A(U)$-essential boundary* of U is the set of points in bdy U which are not $A(U)$-inessential boundary points. For the purposes of this paper, we abbreviate $A(U)$-essential to essential.

If the essential boundary of U is empty, then $A(U)$ consists only of constant functions, and it is immediate that $A(U)$ is complex pervasive on bdy U if and only if Re $A(U)$ is real pervasive on bdy U, i.e. if and only if bdy U has at most two different points.

Let us define the *regularization* of U to be the set

$$\tilde{U} = U \cup \{p \in \text{bdy } U : p \text{ is an essential boundary point of } U\}.$$

We observe that if $\tilde{U} \neq \hat{\mathbb{C}}$ (i.e. if the essential boundary of U is nonempty) then $\hat{\mathbb{C}} \backslash \tilde{U}$ has positive continuous analytic capacity and hence bdy $\tilde{U}$ has positive logarithmic capacity, so harmonic measures exist [2], [5].

PROPOSITION 2.2. *Let $U \subset \hat{\mathbb{C}}$ be open and suppose that the essential boundary of U is nonempty. Let n be the number (possibly infinite) of inessential boundary points of U.*

(i) *If $n \geqslant 1$ then $A(U)$ is not complex pervasive on* bdy U.

(ii) *If $n > 1$ then* Re $A(U)$ *is not real pervasive on* bdy U.

(iii) *If $n = 1$ then* Re $A(U)$ *is real pervasive on* bdy U *if and only if*

(a) *$A(U)$ is Dirichlet on the essential boundary of U, and*

(b) *the component in $\tilde{U}$ of the inessential boundary point of U has boundary equal to the essential boundary of U.*

Proof. (i) Suppose a is an inessential boundary point. Since $A(\tilde{U}) \neq C(\text{bdy } \tilde{U}, \mathbb{C})$, there is a non-trivial annihilating measure on bdy $U \backslash \{a\}$, so $A(U)$ is not complex pervasive.

(ii) Suppose that U has more than one inessential boundary point, and let a and b be two different inessential boundary points of U. Consider for a the harmonic measure λ_a on bdy $\tilde{U}$. Then $\delta_a - \lambda_a \in \mathrm{M}(\text{bdy } U, \mathbb{R})$, where δ_a is the Dirac measure concentrated at a.

It is clear that $\delta_a - \lambda_a \perp \operatorname{Re} A(U)$ and $b \notin \operatorname{spt}(\delta_a - \lambda_a)$ so $\operatorname{Re} A(U)$ is not pervasive, as required.

(iii) Suppose U has only one inessential boundary point, say a.

Suppose $\operatorname{Re} A(U)$ is pervasive on bdy U but $A(U)$ is not Dirichlet on bdy $\tilde{U}$. Then we can choose a nonzero measure $\mu \in \mathrm{M}(\text{bdy } \tilde{U}, \mathbb{R})$, $\mu \perp \operatorname{Re} A(\tilde{U})$ with spt $\mu \subset \tilde{U}$, contradicting the assumption that $\operatorname{Re} A(U)$ is pervasive.

If the boundary of the component in $\tilde{U}$ of a does not coincide with bdy $\tilde{U}$, then $\delta_a - \lambda_a \perp \operatorname{Re} A(U)$ but $\operatorname{spt}(\delta_a - \lambda_a) \subsetneqq \text{bdy } U$ contradicting the fact that $\operatorname{Re} A(U)$ is real pervasive on bdy U.

Conversely, suppose $A(U)$ is Dirichlet on bdy $\tilde{U}$ and the component in $\tilde{U}$ of a has boundary equal to bdy $\tilde{U}$.

Consider a nontrivial real measure $\mu \perp \operatorname{Re} A(U)$ with $\operatorname{spt} \mu \subsetneqq \text{bdy } U$. Clearly $\operatorname{spt} \mu \not\subset \text{bdy } \tilde{U}$ since $A(U)$ is Dirichlet on bdy $\tilde{U}$. So

$$\mu = \alpha\delta_a + v,$$

where $0 \neq \alpha \in \mathbb{R}$ and $v \in \mathrm{M}(\text{bdy } \tilde{U}, \mathbb{R})$. Then

$$\int f d(\alpha\lambda_a + v) = \int f\, d\mu, \qquad \forall f \in A(U),$$

so $\alpha\lambda_a + v \perp A(U)$ and therefore $v = -\alpha\lambda_a$, since $A(U)$ is Dirichlet on bdy $\tilde{U}$.

The support of λ_a is the whole boundary of the component of a in $\tilde{U}$, so is the whole essential boundary. Hence $\operatorname{spt} \mu = \text{bdy } U$, which is impossible. Thus $\operatorname{Re} A(U)$ is real pervasive. ∎

In view of Proposition 2.2 and the Gamelin–Garnett Theorem, we understand pervasiveness when there are inessential boundary points. So it remains to consider the case when the entire boundary of U is essential.

3. THE CONNECTED, ESSENTIAL CASE

Let m be the Lebesgue measure on $\mathbb{C}$. Let μ be a complex measure with compact support. The *Cauchy transform* of μ is defined by

$$\hat{\mu}(\xi) = \frac{1}{\pi} \int \frac{d\mu(z)}{\xi - z}.$$

We denote by $R(K)$ the uniform closure on $\hat{\mathbb{C}}$ of the algebra of all continuous functions on $\hat{\mathbb{C}}$ that are analytic near the compact set K. This coincides, by Runge's Theorem, with the closure of the algebra of all functions continuous on $\hat{\mathbb{C}}$ that coincide near K with some rational function.

The following theorem summarizes well-known results and we state it without proof [1], [5].

THEOREM 3.1. *Let μ be a complex measure with compact support in $\hat{\mathbb{C}}$. Then*

(i) *$\hat{\mu}$ is defined m-almost everywhere, i.e. $|\hat{\mu}(z)| < \infty$ for almost all $z \in \mathbb{C}$.*

(ii) *$\hat{\mu}$ is holomorphic on $\mathbb{C} \backslash \operatorname{spt} \mu$.*

(iii) *If $\hat{\mu} = 0$ m-almost everywhere, then $\mu = 0$.*

(iv) *Let $K \subset \hat{\mathbb{C}}$ be a compact set. Then $\hat{\mu}$ vanishes off K if and only if $\mu \perp R(K)$.*

(v) *If $K \subset \mathbb{C}$ is compact and $\mu = m_{|K}$, where $m_{|K}$ stands for the restriction of the Lebesgue measure to K, then $\hat{\mu}$ is continuous.*

THEOREM 3.2. *Let U be a connected open subset of $\hat{\mathbb{C}}$, and let* bdy U *be nonempty and essential. Then $A(U)$ is complex pervasive on* bdy U. *A fortiori,* Re $A(U)$ *is real pervasive on* bdy U.

Proof. Let $\mu \in \mathrm{M}(\operatorname{bdy} U, \mathbb{C})$, $\mu \perp A(U)$ and suppose that $\operatorname{spt} \mu \neq \operatorname{bdy} U$. We shall prove that $\mu = 0$.

As $\mu \perp A(U)$, it follows that $\mu \perp R(\operatorname{clos} U)$ so by (iv) of Theorem 3.1, $\hat{\mu} = 0$ in $\hat{\mathbb{C}} \backslash \operatorname{clos} U$.

Suppose now that $a \in \operatorname{bdy} U \backslash \operatorname{spt} \mu$, $a \neq \infty$. Choose $r > 0$ sufficiently small so that $\mathbb{B}(a, r) \cap \operatorname{spt} \mu = \emptyset$, where $\mathbb{B}(a, r)$ denotes the closed ball with centre a and radius r. Let $\{r_n\}_{n-1}^{\infty}$ be a sequence of positive real numbers so that $r_n < r$ and $r_n \downarrow 0$ as $n \uparrow \infty$. By hypothesis, there exist compact sets $K_n \subset \mathbb{B}(a_n, r_n) \backslash U$ having positive continuous analytic capacity $\alpha(K_n)$, so there exists $f_n \in A(\hat{\mathbb{C}} \backslash K_n)$, f_n nonconstant and $\sup_{\hat{\mathbb{C}}} |f_n| = 1$. Multiply f_n, if need be, by an unimodular constant, so that we obtain $f_n(p_n) = 1$ for some $p_n \in K_n$ and $|f_n| < 1$ off $\mathbb{B}(a_n, r_n)$ by the maximum modulus principle. Note that $\hat{\mu}$ is analytic near $\mathbb{B}(a, r)$.

Next, $\hat{\mu}(p_n) = 0$ because otherwise the measure ν defined by

$$d\nu(z) = -\frac{1}{\pi} \frac{1}{\hat{\mu}(p_n)} \frac{d\mu(z)}{z - p_n}$$

is a complex representing measure for p_n on $A(U)$ and

$$1 = f_n^k(p_n) = \int f_n^k \, d\nu \to 0 \qquad \text{as} \quad k \uparrow +\infty$$

which is a contradiction.

Consequently, $\hat{\mu}(a) = 0$ by continuity. Since a is an essential boundary point of U, it follows that $\mathbb{B}(a, r) \cup \text{bdy}\, U$ is uncountable. The previous argument shows that $\hat{\mu} = 0$ on $\mathbb{B}(a, r) \cap \text{bdy}\, U$. By (ii) of Theorem 3.1, $\hat{\mu}$ is analytic on $\mathbb{C} \backslash \text{spt}\, \mu$, so $\hat{\mu} = 0$ on $\mathbb{B}(a, r)$, and therefore, since U is connected, $\hat{\mu} = 0$ on U. Hence $\hat{\mu} = 0$ on $\hat{\mathbb{C}} \backslash \text{spt}\, \mu$.

Finally, let $E \subset \text{bdy}\, U$ be compact. Let $\lambda = m_{|E}$. By (v) of Theorem 3.1, $\hat{\lambda}$ is continuous and therefore $\hat{\lambda} \in A(U)$, so by Fubini's Theorem

$$0 = \int \hat{\lambda} \, d\mu = -\int \hat{\mu} \, d\lambda = \int_E \hat{\lambda} \, dm,$$

so $\hat{\mu} = 0$ m-almost everywhere on bdy U.

As $\text{spt}\, \mu \in \text{bdy}\, U$ it follows then that $\hat{\mu} = 0$ m-almost everywhere on $\hat{\mathbb{C}}$, so by (iii) of Theorem 3.1, $\mu = 0$. ∎

4. MULTIPLE COMPONENTS

We deal first with complex pervasiveness.

THEOREM 4.1. *Suppose U is a (possibly disconnected) proper open subset of $\hat{\mathbb{C}}$ without inessential boundary points. Then $A(U)$ is complex pervasive on* bdy U *if and only if* bdy $U_i =$ bdy U *for each component U_i of U.*

Proof. The "if" direction is proved by essentially the same argument as that for Theorem 3.2.

To see the "only if" direction, suppose U has a component U_i with bdy $U_i \neq$ bdy U. We may choose a nonzero annihilating measure μ for $A(U_i)$ supported on bdy U_i, which is a proper subset of bdy U. Then μ annihilates $A(U)$, and this shows that $A(U)$ is not complex pervasive on bdy U. ∎

Remark 4.2. The vagaries of plane topology allow up to an infinite number of connected open sets to share a common boundary.

Moving on to real pervasiveness, we note first:

THEOREM 4.3. *Suppose $U \subset \hat{\mathbb{C}}$ is open and proper, with no inessential boundary points. Suppose U is not connected, and* Re $A(U)$ *is real pervasive* *on* bdy U. *Then* U *has at most one component that is not simply-connected. Furthermore, if* U *has such a component* U_k, *then* bdy $U_k =$ bdy U.

Remark 4.4. If $U \subset \hat{\mathbb{C}}$ is open, not connected, and several components of U have boundary equal to bdy U, then all components of U are simply-connected.

For suppose U_i and U_k are different components of U, and bdy $U_k =$ bdy U. Then U_i is one of the components of $\hat{\mathbb{C}} \backslash \mathrm{clos}\, U_k$, which is the complement of a continuum, and hence U_i is simply-connected.

Proof of Theorem 4.3. Suppose that Re $A(U)$ is real pervasive and let U_i be a component of U so that bdy $U_i \neq$ bdy U.

Clearly, Re $A(U) \subset$ Re $A(U_i)$. Therefore the restriction of Re $A(U)$ to bdy U_i, Re $A(U)_{|\mathrm{bdy}\, U_i}$, is dense in $C(\mathrm{bdy}\, U_i, \mathbb{R})$. Hence $A(U_i)$ is a Dirichlet algebra on bdy U_i, so we can conclude that U_i is simply-connected [6].

Suppose next that U has at least two different components U_k, U_l, that are not simply-connected. Then from the foregoing bdy $U_k =$ bdy $U_l =$ bdy U. Hence U_k and U_l are both components of $\hat{\mathbb{C}} \backslash \mathrm{bdy}\, U$ and by Remark 4.4, both are simply connected, a contradiction. ∎

In the other direction we have

THEOREM 4.5. *Suppose* $U \subset \hat{\mathbb{C}}$ *is open and proper, with no inessential boundary points. Suppose* U *has at least one component* U_k *so that* bdy $U_k =$ bdy U. *Then* Re $A(U)$ *is real pervasive on* bdy U.

The proof of this theorem involves the theory of T-invariant algebras. We review the basic notation and ideas.

For a continuous function $f \in C(\mathbb{C}, \mathbb{C})$, having compact support, we define the Cauchy transform

$$Cf = \widehat{fm},$$

where m, as before, denotes the Lebesgue measure on $\mathbb{C}$. We have

$$\frac{\partial}{\partial \bar{z}}(Cf) = f$$

in the sense of distributions, so that (by Weyl's Lemma), Cf is holomorphic off spt f.

For $\varphi \in C^{\infty}_{\mathrm{cs}}(\mathbb{C}, \mathbb{C})$ (the space of infinitely differentiable functions having compact support) and $f \in C(\hat{\mathbb{C}}, \mathbb{C})$, we define

$$T_{\varphi} f = \varphi f - C\left(f \frac{\partial \varphi}{\partial \bar{z}}\right).$$

The linear operator T_φ (the *Vitushkin localization operator*) is continuous from $C(\hat{\mathbb{C}}, \mathbb{C})$ into itself.

A subalgebra $A \subset C(\hat{\mathbb{C}}, \mathbb{C})$ is said to be *T-invariant* if

$$T_\varphi f \in A, \qquad \forall f \in A, \qquad \forall \varphi \in C^\infty_{\mathrm{cs}}(\mathbb{C}, \mathbb{C}).$$

We note that

$$\frac{\partial}{\partial \bar{z}} T_\varphi f = \varphi \frac{\partial f}{\partial \bar{z}}$$

in the sense of distributions, so that $T_\varphi f$ is holomorphic whenever f is holomorphic and off spt φ. This is the basis for the utility of T_φ in localizing singularities of analytic functions. It is obvious from this observation that $A(U)$ is a T-invariant algebra, whenever $U \subset \hat{\mathbb{C}}$ is open. So also is $\mathcal{O}(E)$, the algebra of all functions continuous on $\hat{\mathbb{C}}$ and holomorphic near E, whenever $E \subset \hat{\mathbb{C}}$. Since T_φ is continuous on $C(\hat{\mathbb{C}}, \mathbb{C})$ it follows that $\mathrm{clos}_{C(\hat{\mathbb{C}}, \mathbb{C})} \mathcal{O}(K)$ is also T-invariant. But this closure is, by Runge's Theorem, equal to $R(K)$, whenever K is compact.

LEMMA 4.6. *Let $U \subset \hat{\mathbb{C}}$ be open and $K \subset \mathbb{C}$ compact. Then*

$$B = \mathrm{clos}_{C(\hat{\mathbb{C}}, \mathbb{C})}(A(U) + R(K))$$

is a T-invariant algebra.

Proof. It is obvious that $A(U) + R(K)$ is T-invariant, and hence so is B. Also

$$B = \mathrm{clos}_{C(\hat{\mathbb{C}}, \mathbb{C})}(A(U) + \mathcal{O}(K)),$$

so it suffices to show that $A(U) + \mathcal{O}(K)$ is an algebra, i.e. to show that if $f_1, f_2 \in A(U) + \mathcal{O}(K)$, then $f_1 f_2 \in A(U) + \mathcal{O}(K)$.

Fix $f_1, f_2 \in A(U) + \mathcal{O}(K)$, and choose $g_i \in A(U), h_i \in \mathcal{O}(K)$ such that $f_i = g_i + h_i$, for $i = 1, 2$. Then

$$f_1 f_2 = g_1 g_2 + g_1 h_2 + g_2 h_1 + h_1 h_2,$$

so it suffices to show that $g_1 h_2$ and $g_2 h_1$ belong to $A(U) + \mathcal{O}(K)$.

So let $g \in A(U)$ and $h \in \mathcal{O}(K)$. We need to show that $gh \in A(U) + \mathcal{O}(K)$. Choose an open set $W \supset K$, such that h is holomorphic on W.

Pick $\varphi \in C^\infty_{\mathrm{cs}}(\mathbb{C}, \mathbb{C})$ so that $\varphi = 1$ near K and $\varphi = 0$ off W. Then $1 - \varphi = 0$ near K and $1 - \varphi = 1$ off W.

Let $u = T_\varphi gh$ and $v = gh - T_\varphi gh$. Then

$$\frac{\partial u}{\partial \bar{z}} = \varphi g \frac{\partial h}{\partial \bar{z}} + \varphi h \frac{\partial g}{\partial \bar{z}}$$

$$\frac{\partial v}{\partial \bar{z}} = (1-\varphi)\, g \frac{\partial h}{\partial \bar{z}} + (1-\varphi)\, h \frac{\partial g}{\partial \bar{z}}.$$

Thus $u \in A(U)$ and $v \in \mathcal{O}(K)$, and $gh = u + v$, so we are done. ∎

It is possible to associate a capacity γ_A to each T-invariant algebra A [4], generalizing the association of $E \mapsto \alpha(E \setminus U)$ to $A(U)$. Davie showed that closed T-invariant algebras are uniquely determined by their corresponding capacities.

Davie's Theorem [4, Theorem 2.3, p. 414]. *Let A_0 denote the algebra of all bounded Borel functions on $\mathbb{C}$ which are analytic outside some compact set. Let A_1 and A_2 be T-invariant subalgebras of A_0, and suppose all functions in A_1 are continuous on $\mathbb{C}$. Suppose also that for all $z \in \mathbb{C}$ we can find $m, r, \delta_0 > 0$ with $\gamma_{A_1}(\mathbb{U}(z, \delta)) \leqslant m\gamma_{A_2}(\mathbb{U}(z, r\delta))$ for $0 < \delta < \delta_0$. Let $f \in A_1$. Then f is in the uniform closure of A_2.*

We note that the result also holds if $A_0 \cap A_i$ is a uniformly dense subset of A_i ($i = 1$ and, 2). This will be the case in our application of the approximation lemma below.

Lemma 4.7. *Let $U \subset \hat{\mathbb{C}}$ be open and proper, and suppose* bdy *U is essential. Let $\{U_i : i \in I\}$ be the set of connected components of U. Let $a \in$* bdy *U and $r > 0$. Let*

$$V = \mathbb{U}(a, r) \cup \bigcup_{i \in I} \{U_i : U_i \cap \mathbb{U}(a, r) \neq \varnothing\}$$

$$K = \hat{\mathbb{C}} \setminus V$$

$$W = \bigcup_{i \in I} \{U_i : U_i \cap \mathbb{U}(a, r) = \varnothing\},$$

and

$$B = A(U) + R(K).$$

Then B is dense in $A(W)$.

Proof. By Davie's Theorem, it suffices to show that there exist $m \geqslant 1$, $t \geqslant 1$, $\delta_0 > 0$ such that

$$\alpha(\mathbb{U}(z, \delta)\backslash U) \leqslant m\gamma_B(\mathbb{U}(z, t\delta))$$

whenever $z \in \mathbb{C}$, $0 < \delta < \delta_0$.

We will show that $m = 4$, $t = 2$, $\delta_0 = r$ work.

First we note

$$\begin{aligned}\gamma_B(\mathbb{U}(z, \delta)) &\geqslant \max\{\gamma_{A(U)}(\mathbb{U}(z, \delta)), \gamma_{R(K)}(\mathbb{U}(z, \delta))\} \\ &= \max\{\alpha(\mathbb{U}(z, \delta)\backslash U), \alpha(\mathbb{U}(z, \delta)\backslash K)\}.\end{aligned}$$

Fix $z \in \mathbb{C}$ and δ, with $0 < \delta < r$. There are two cases.

(i) $\operatorname{dist}(z, V) < \delta$. Then $\mathbb{U}(z, 2\delta)\backslash K$ contains an arc $\beta \subset V$ of diameter at least δ, so [5, p. 199, 203]

$$\begin{aligned}\alpha(\mathbb{U}(z, \delta)\backslash W) &\leqslant \alpha(\mathbb{U}(z, \delta))(= \delta) \\ &\leqslant 4\alpha(\mathbb{U}(z, 2\delta)\backslash K) \leqslant 4\gamma_B(\mathbb{U}(z, 2\delta)).\end{aligned}$$

(ii) $\delta \leqslant \operatorname{dist}(z, V)$. Then $\mathbb{U}(z, \delta) \subset K$, so

$$\alpha(\mathbb{U}(z, \delta)\backslash W) = \alpha(\mathbb{U}(z, \delta)\backslash U) \leqslant \gamma_B(\mathbb{U}(z, \delta)) \leqslant 4\gamma_B(\mathbb{U}(z, 2\delta)).$$

So the result follows. ∎

Proof of Theorem 4.5. In view of Theorem 3.2, we may assume that U has multiple components.

Let U_k be one component having bdy $U_k =$ bdy U.

Let $\mu \in \mathrm{M}(\text{bdy } U, \mathbb{R})$, $\mu \perp A(U)$, and spt $\mu \subsetneqq$ bdy U. We wish to show that $\mu = 0$.

Choose $a \in$ bdy U, $r > 0$ such that $\mathbb{B}(a, r) \cap \operatorname{spt} \mu = 0$.

Let V, K, W and B be constructed as in the statement of Lemma 4.7. We note that $U_k \subset V$. Also each component of W is simply-connected.

The argument of Theorem 3.2 tells us that $\hat{\mu} = 0$ on $\mathbb{U}(a, r)$, and hence on V, so that $\mu \perp R(K)$. Thus $\mu \perp B$, hence $\mu \perp A(W)$, by Lemma 4.7.

The facts that $U_k \cap W = \varnothing$, and that each boundary point of W is also a boundary point of U_k, tell us that there are no inner boundary points for W and hence no GG-points for W on bdy W. Thus, by the Gamelin–Garnett Theorem, $A(W)$ is Dirichlet on bdy W. But μ is a real measure, so $\mu = 0$. ∎

Thus we have completely solved the question of when $A(U)$ is real pervasive except in the base when all components of U are simply-connected, and no component U_i of U has bdy $U_i =$ bdy U. To deal with this, we introduce some terminology.

DEFINITION 4.8. We say that a point $p \in \text{bdy}\, U$ *influences* $q \in \text{bdy}\, U$ (with respect to U) if for all $r > 0$ there exists U_i, a component of U, such that

$$\mathbb{U}(p, r) \cap U_i \neq \varnothing \qquad \text{and} \qquad \mathbb{U}(q, r) \cap U_i \neq \varnothing.$$

Remark 4.9. Note that the relation $\{(p, q) \in \text{bdy}\, U \times \text{bdy}\, U \colon p \text{ influences } q\}$ is reflexive and symmetric.

THEOREM 4.10. *Let $U \subset \hat{\mathbb{C}}$ be open and proper, with no inessential boundary points. Suppose all components of U are simply-connected and no component has* bdy $U_i =$ bdy U. *Then the following statements are equivalent.*

(i) Re $A(U)$ *is real pervasive on* bdy U.

(ii) *For every $p \in$* bdy U, *and for every GG-point q for U on* bdy U, *p influences q.*

LEMMA 4.11. *Suppose U, a, r, V, K and W are as in Lemma* 4.7. *Suppose that all components of W are simply-connected. Then there are no GG-points for U on* bdy $W \setminus$bdy V *if and only if $A(W)$ is Dirichlet on* bdy W.

Proof. To prove the "only if" direction, by the Gamelin–Garnett Theorem it suffices to show that there are no GG-points for W on bdy W.

Let z be a boundary point of W.

If $z \in \text{bdy}\, W \setminus \text{bdy}\, V$, then there exists $\delta > 0$ such that $\mathbb{U}(z, \delta) \cap W = \mathbb{U}(z, \delta) \cap U$, so z is not a GG-point for W.

If $z \in \text{bdy}\, V$, then for $0 < \delta < r$, $\mathbb{U}(z, \delta) \cap V$ contains an arc of diameter $\delta/2$, hence $\alpha(\mathbb{U}(z, \delta) \setminus W) \geqslant \delta/8$. Thus z is not a GG-point for W.

The "if" direction is clear from the fact that if $A(W)$ is Dirichlet on bdy W then there are no GG-points for W on bdy W, so neither are there any for U. ∎

Proof of Theorem 4.10. An argument similar to the proof of Theorem 4.5 shows that (ii) implies (i).

To see that (i) implies (ii), suppose (ii) fails.

Pick $p \in \text{bdy}\, U$, $q \in \text{bdy}\, U$, $r > 0$ such that q is a GG-point for U and $\mathbb{U}(p, r) \cap U_i \neq \varnothing$ implies $\mathbb{U}(q, r) \cap U_i = \varnothing$, whenever U_i is a connected component of U.

Let V, K, and W be constructed as in Lemma 4.7, with a replaced by p. Then by Lemma 4.11, $A(W)$ is not Dirichlet on bdy W, so there exists a real measure μ, supported on bdy W, annihilating $A(W)$. Since bdy $W \subsetneqq$ bdy U and $A(U) \subset A(W)$, this shows that Re $A(U)$ is not real pervasive on bdy U. ▮

We close with some examples, and a question.

EXAMPLE 4.12. Let $a_n \in \mathbb{R}$ and $r_n > 0$ such that the intervals $[a_n - r_n, a_n + r_n]$ are pairwise-disjoint and $\bigcup_{n=1}^{\infty} [a_n - r_n, a_n + r_n]$ is dense in $\mathbb{R}$. Let

$$T = \mathbb{R} \cup \bigcup_{n=1}^{\infty} \mathbb{B}(a_n, r_n)$$

$$U = \mathbb{C} \setminus T.$$

Then U is open and has two components, U_1 and U_2. We can arrange that the r_n are so small that $\mathbb{R}$ has GG-points for U. For instance, 0 will be a GG-point for U if

$$\sum_{|a_n| < r} r_n < r^2, \qquad \forall r > 0.$$

In that case, $A(U)$ is not Dirichlet on bdy U, but Re $A(U)$ is real pervasive on bdy U, by Theorem 4.10, since all GG-points lie on $\mathbb{R}$ and are influenced by each boundary point of U. Theorem 4.1 tells us that $A(U)$ is not complex pervasive on bdy U.

EXAMPLE 4.13. If we modify Example 4.12 so that $\bigcup_{n=1}^{\infty} [a_n - r_n, a_n + r_n]$ has for its closure $[-2, -1] \cup [1, 2]$ and take

$$T = [-2, -1] \cup [1, 2] \cup \bigcup_{n=1}^{\infty} \mathbb{B}(a_n, r_n)$$

$$U = \mathbb{C} \setminus T.$$

Then U is connected, so $A(U)$ is complex pervasive on bdy U by Theorem 3.2, but U is not simply-connected, so $A(U)$ is not Dirichlet on bdy U.

EXAMPLE 4.14. Let T be as in Example 4.12 and let

$$S = T \cup \{iz \colon z \in T\}.$$

Then we can arrange that $U = \mathbb{C} \setminus S$ has four components and there are GG-points for U on the positive and negative real and imaginary axes. In that case, for each point $p \in$ bdy U there exists a GG-point q not influenced by p. Thus $A(U)$ is not real pervasive on bdy U.

Question. It is possible to find an open set $U \in \hat{\mathbb{C}}$ such that each connected component is simply-connected, no component U_i has bdy $U_i = \text{bdy } U$, but each boundary point influences all the others. We do not know whether there is such U having GG-points, i.e. for which $A(U)$ is not Dirichlet. Is this possible?

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